Part III - Lognormally-Distributed Asset Values

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September, 2018

In Part I of this series we will built an asset value model that assumed that asset values were normally-distributed. In Part III we will build an asset value model that assumes that asset values are lognormally-distributed. To that end we will use the hypothetical problem from Part I, which is...

Our Hypothetical Problem

We are tasked with building an asset value model that assumes that asset values are normally-distributed. We are given the following go-forward model parameters...

Table 1: Portfolio Composition

Description	Value
Asset value at time zero (in dollars)	100.00
Annual return mean	12.00%
Annual distribution rate	5.00%
Annual return volatility	30.00%

Our task is to answer the following questions:

Question 1: What is expected asset value at the end of year three?

Question 2: What is the probability that asset value will be less than $80.00 at the end of year three?

Asset Value

In Part I we defined the variable $A(0)$ to be asset value at time zero, the variable m to be the expected rate of return, which is the the expected annual rate of return minus the expected annual distribution rate, and the variable v to be return variance. Using the parameters in Table 1 above the equation for normally-distributed asset value is...

$$A(t) = A(0)\left(1+m\right)^{t}\left(1+\sqrt{v\,t}\,Z\right) \text{ ...where... } Z \sim N\left[0,1\right] \tag{1}$$

Note that asset value in Equation (1) above uses annual compounding. If the variable n is the number of annual compounding periods then we can rewrite Equation (1) above as...

$$A(t) = A(0)\left(1+\frac{m}{n}\right)^{t\,n}\left(1+\sqrt{v\,t}\,Z\right) \text{ ...where... } Z \sim N\left[0,1\right] \tag{2}$$

As the number of annual compounding periods goes to infinity then we can rewrite the components of asset value Equation (2) above as...

$$\lim_{n\to\infty}\left(1+\frac{m}{n}\right)^{t\,n} = \text{Exp}\left\{m\,t\right\} \text{ ...and... } \left(1+\sqrt{v\,t}\,Z\right) = \text{Exp}\left\{\sqrt{v\,t}\,Z\right\} \tag{3}$$

Using Equation (3) above we can rewrite asset value Equation (2) above as the following continuous-time equation...

$$A(t) = A(0)\,\text{Exp}\left\{\lambda\,t + \sigma\,\sqrt{t}\,Z\right\} \text{ ...where... } \lambda = \text{return mean} \text{ ...and... } \sigma = \text{return volatility} \tag{4}$$

Note that asset value per Equation (1) above is normally distributed and asset value per Equation (4) above is lognormally distributed. When we change the distribution of asset value we don't want that change to add or subtract value. This statement in equation form is...

$$\text{We want...}\ \mathbb{E}\bigg[A(0)\Big(1+m\Big)^t\Big(1+\sqrt{v\,t}\,Z\Big)\bigg]=\mathbb{E}\bigg[A(0)\,\text{Exp}\Big\{\lambda\,t+\sigma\,\sqrt{t}\,Z\Big\}\bigg] \tag{5}$$

We will define the variable λ to be the continuous time return mean and the variable σ to be the continuous-time return volatility. If we define these two variables as follows then the condition in Equation (5) above will be met...

$$\lambda=\ln\Big(1+m\Big)-\frac{1}{2}\,\sigma^2\ \ \text{...and...}\ \ \sigma=\sqrt{v} \tag{6}$$

Since asset value in continuous-time is lognormally-distributed then using Equation (6) above the equation for the expected value of Equation (4) above is...

$$\mathbb{E}\Big[A(t)\Big]=\mathbb{E}\Big[A(0)\Big(1+m\Big)^t\Big]=A(0)\,\text{Exp}\bigg\{\Big(\lambda+\frac{1}{2}\,\sigma^2\Big)t\bigg\} \tag{7}$$

Using Appendix Equation (13) below the probability that actual asset value at time t will be less than some threshold value is...

$$\text{Prob}\Big[\text{Actual value} < \text{Threshold value}\Big]=\text{NORMSDIST}(Z)\ \ \text{...where...}\ \ Z=\bigg[\ln\bigg(\frac{A(t)}{A(0)}\bigg)-\lambda\,t\bigg]\bigg/\sigma\sqrt{t} \tag{8}$$

The Solution To Our Hypothetical Problem

Using Equation (6) above and the parameters in Table 1 above the values of model parameters lambda and sigma are...

$$\lambda=\ln\Big(1+0.12-0.05\Big)-\frac{1}{2}\times 0.3000^2=0.0227\ \ \text{...and...}\ \ \sigma=0.3000 \tag{9}$$

Question 1: What is expected asset value at the end of year three?

Using Equations (7) and (9) above and the parameters in Table 1 above the answer to question one is...

$$\mathbb{E}\Big[A(3)\Big]=100.00\times\text{Exp}\bigg\{\Big(0.0227+\frac{1}{2}\times 0.3000^2\Big)\times 3\bigg\}=122.50 \tag{10}$$

Question 2: What is the probability that asset value will be less than \$80.00 at the end of year three?

Using Equations (8) and (9) above and the parameters in Table 1 above the answer to question two is...

$$\begin{aligned}&\text{Prob}\Big[\text{Actual value} < 80.00\Big]=\text{NORMSDIST}(Z)=0.2876\\&\text{where...}\ \ Z=\bigg[\ln\bigg(\frac{80.00}{100.00}\bigg)-0.0227\times 3\bigg]\bigg/0.3000\times\sqrt{3}=-0.5604\end{aligned} \tag{11}$$

Appendix

A. Using Equation (4) above and solving for the random variable Z...

$$\begin{aligned}A(t)&=A(0)\,\text{Exp}\Big\{\lambda\,t+\sigma\,\sqrt{t}\,Z\Big\}\\\ln\bigg(\frac{A(t)}{A(0)}\bigg)&=\lambda\,t+\sigma\,\sqrt{t}\,Z\\Z&=\bigg[\ln\bigg(\frac{A(t)}{A(0)}\bigg)-\lambda\,t\bigg]\bigg/\sigma\sqrt{t}\end{aligned} \tag{12}$$

Using Equation (12) above the probability that actual asset value at time t will be less than the threshold value $A(t)$ is...

$$\text{Prob}\Big[\text{Actual value at time } t < A(t)\Big]=\text{NORMSDIST}(Z) \tag{13}$$